AN END-TO-END CONSTRUCTION OF DOUBLY PERIODIC MINIMAL SURFACES

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Abstract. Using Traizet's regeneration method, we prove that for each positive integer n there is a family of embedded, doubly periodic minimal surfaces with parallel ends in Euclidean space of genus $2n-1$ and 4 ends in the quotient by the maximal group of translations. The genus $2n-1$ family converges smoothly to $2n$ copies of Scherk's doubly periodic minimal surface. The only previously known doubly periodic minimal surfaces with parallel ends and genus greater than 3 limit in a foliation of Euclidean space by parallel planes.

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1. Introduction

A *doubly periodic* minimal surface M in $\mathbb{R}^3$ is invariant under two linearly independent translations. The first such surface was discovered by Scherk in 1835 [11].

Let Λ be the 2-dimensional lattice generated by the translation vectors, which we assume are horizontal. If the quotient surface M/Λ is complete, properly embedded, and of finite topology then Meeks and Rosenberg [8] proved that it has a finite number of top and bottom ends, which are asymptotic to flat annuli. These ends are referred to as Scherk ends. There must be an even number of top ends and an even number of bottom ends. In order for the surface to be embedded, the top ends must be parallel to each other and the bottom ends must also be parallel to each other.

Either the top and bottom ends are parallel or non-parallel. By results of Hauswirth and Traizet [4], embedded doubly periodic minimal surfaces that are nondegenerate lie in a moduli space of dimensions 3 in the parallel case and 1 in the non-parallel case. Fix the direction and length of one of the horizontal period vectors. In the parallel case, the parameters are given by the length of the other period vector, the angle between the period vectors, and the angle between the vertical axis and the flux vectors at the ends. In the non-parallel case, the parameter is given by the angle between the period vectors. In this case, Meeks and Rosenberg [8] proved that the period vectors must have the same length if the number of top and bottom ends are the same, and it is easy to show that the flux vectors must be vertical.

As was shown by Lazard-Holly and Meeks [6], Scherk's surface is the only embedded doubly periodic minimal surface of genus zero. They lie in a one-parameter family, with the parameter giving the angle $\theta \in (0, \pi/2]$ between the top and bottom ends.

In 1988, Karcher [5] and Meeks and Rosenberg [8] proved the existence of one-parameter families of genus one, embedded examples that were shown to lie in a three-dimensional family by Rodríguez [10]. In fact, Pérez, Rodríguez, and Traizet [9] proved that they are the only genus one doubly periodic embedded minimal surfaces with parallel ends. We refer to these as KMR surfaces.

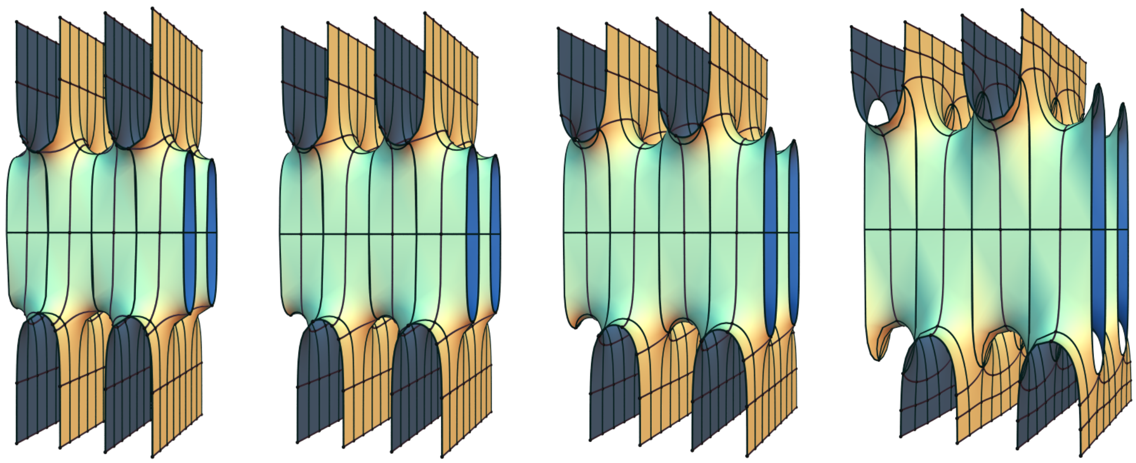

FIGURE 1.1. Members of the KMR family near its limit as two doubly periodic Scherk surfaces.

Rodríguez [10] showed, near one boundary of their moduli space, KMR surfaces converge smoothly to two copies of Scherk's doubly periodic minimal surface. In Figure 1.1, the ends of two copies of the doubly periodic Scherk surface appear to be glued with various translations. In our construction, we will realize this translation as Dehn twists controlled by $v_{k,i}$. See section 2.

This led to the conjecture that there exist surfaces that converge to more than two copies of Scherk's doubly periodic surface. We prove the existence of surfaces that converge to an even number of copies of Scherk's doubly periodic surface. In this case, the surfaces have parallel ends and thus live in a 3-parameter family. The construction does not work for surfaces with non-parallel ends which would converge to an odd number of copies of Scherk's doubly periodic surface. In this case, the surfaces would live in a 1-parameter family, and the construction we use requires two free parameters.

Theorem 1.1. *For each even positive integer n and angle $\theta \in (0, \pi/2)$, there exists a two-parameter family of genus $2n-1$, embedded doubly periodic minimal surfaces with parallel ends that converges smoothly to $2n$ spheres joined by nodes such that the Weierstrass data on each sphere converges to the Weierstrass data of Scherk's doubly periodic minimal surface, with angle θ between the top and bottom ends of each copy.*

In what follows, we will obtain the underlying noded Riemann surface by gluing $2n$ spheres, $\overline{\mathbb{C}}_k$. If we fix the base point of the Weierstrass representation in the sphere $\overline{\mathbb{C}}_k$, then the limiting object will be the k-th doubly periodic Scherk surface while the other Scherk surfaces disappear to infinity. Additionally, we will obtain a Dehn twist between each pair of doubly periodic Scherk surfaces, but the Dehn twists need not all be the same.

Note that when $n = 1$, the angle θ can equal $\pi/2$. It is surprising that this isn't the case when $n > 1$. The examples in Figure 1.1 all have $n = 1$ and $\theta = \pi/2$.

We prove Theorem 1.1 by applying Traizet's regeneration technique, adapting the methods used in [13]. There, Traizet constructs triply periodic minimal surfaces that limit in a foliation of $\mathbb{R}^3$ by horizontal planes. Near the limit, the surfaces are asymptotic to horizontal planes joined by catenoidal necks. At the limit, the necks shrink to nodes, and the quotient surface is a noded Riemann surface. The components of this noded surface are punctured tori, and the location of the punctures must satisfy a set of balance equations. Traizet's regeneration technique starts with a noded Riemann surface whose components are punctured tori, with the punctures satisfying a set of balance equations. Then, a space of Riemann surfaces is constructed in a neighborhood of the noded surface, with accompanying Weierstrass data. The period problem is solved using the implicit function theorem.

Traizet's regeneration technique was used by Connor and Weber in [1] to prove that for any genus $g \geq 1$ and any even number $N \geq 2$ there are 3-dimensional families of embedded, doubly periodic minimal surfaces with parallel ends in Euclidean space of genus g and N top and N bottom ends in the quotient by the maximal group of translations. They limit in a foliation of parallel planes in a similar manner as the triply periodic surfaces in [13].

As in [13], we use the most general form of the Weierstrass representation. Suppose Σ is a Riemann surface with punctures $p_1, p_2, \ldots, p_m$, and ϕ_1, ϕ_2, ϕ_3 are meromorphic one-forms on Σ. The map

$$f(z) = \operatorname{Re} \int^z (\phi_1, \phi_2, \phi_3)$$

is a doubly periodic minimal immersion if the following conditions hold:

(1) For any cycle $\gamma \in \mathrm{H}_1(\Sigma)$,

$$\mathrm{Re}\int_\gamma (\phi_1, \phi_2, \phi_3) \in \Lambda$$

(2) For each puncture puncture p_k,

$$\mathrm{Re}\int_{C(p_k)} (\phi_1, \phi_2, \phi_3) \in \Lambda$$

where $C(p_k)$ is a cycle around p_k. Together, (1) and (2) are called the period problem, and they ensure that f is well-defined.

(3) If $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$ then f is conformal and thus minimal, since f is also a harmonic map.

(4) If $|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 > 0$ then f is a regular immersion.

Here, the surfaces Σ are defined by opening the nodes of a noded Riemann surface, and (ϕ_1, ϕ_2, ϕ_3) are defined by specifying the desired periods. The period problem, conformality, and regularity are then solved using the implicit function theorem.

Usually, the location of the nodes must satisfy a complicated set of balance equations. That isn't the case here. We are gluing doubly periodic Scherk surfaces together. Hence, each component of the noded surface will be a Riemann sphere with four punctures, with the location of three of the punctures fixed at 0, 1, and ∞. Each puncture corresponds to a Scherk end.

It is also interesting to note that the triply-periodic Schwarz CLP surface of genus 3 [12] appears to contain the KMR surfaces if we glue the top ends to the bottom ends. This was proven by Ejiri, Fujimori, and Shoda in 2015 [2]. Increasing the number of fundamental pieces taken produces surfaces with odd higher genus where each successive pair of doubly-periodic Scherks is joined by an alternating pattern of Dehn twists. It turns out that we can use any finite pattern of Dehn twists to create triply periodic minimal surfaces of odd genus $g \geq 3$.

Theorem 1.2. *For each positive integer n and angle $\theta \in (0, \pi/2)$ there exists a five-dimensional family of genus $2n+1$, embedded triply periodic minimal surfaces that converges smoothly to $2n$ spheres joined by nodes, where the Weierstrass data on each sphere converges to the Weierstrass data of Scherk's doubly periodic minimal surface, with angle θ between the top and bottom ends of each copy.*

The paper is organized as follows. In section 2, we define the Riemann surfaces Σ_t. Section 3 covers several lemmas on differentials needed to solve the conformal equations and the period problem. We define the Weierstrass data in section 4. The conformal equations and period problem are solved in sections 5 and 6. In section 7, we prove embeddedness and regularity. In section 8, we sketch the proof of Theorem 1.2, which uses the same technique as in the proof of Theorem 1.1.

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2. Riemann surfaces Σ_t

Our goal is to come up with a multi-dimensional family of Riemann surfaces and accompanying Weierstrass data that limits in such a way that we can apply the implicit function theorem to prove there is a two-parameter family of minimal surfaces satisfying the following properties.

- Each surface is an embedded doubly periodic minimal surface with parallel ends.
- The surfaces converge to $2n$ doubly periodic Scherk surfaces glued together.

We use the techniques of Fay [3] to create a family of Riemann surfaces in a neighborhood of a noded Riemann surface. Take $2n$ copies of the Riemann sphere $\overline{\mathbb{C}}$, labeled $\overline{\mathbb{C}}_k$, $k = 1, \dots, 2n$. Label the points 0 and ∞ in each copy as 0_k and ∞_k. There will be special points in each $\overline{\mathbb{C}}_k$ where the ends of the surface will be located or where we will glue together the different copies of $\overline{\mathbb{C}}$. In $\overline{\mathbb{C}}_1$, the points $a_{1,1}$ and $a_{1,2}$ will be where the two gluings will take place and the points 0_1 and ∞_1 will be where the two of the ends are located. In $\overline{\mathbb{C}}_{2k}$ for $k = 1, 2, \dots, n-1$, the gluings will take place at $b_{2k-1,1}, b_{2k-1,2}$, 0_{2k}, and ∞_{2k}. In $\overline{\mathbb{C}}_{2k-1}$ for $k = 2, 3, \dots, n$, the gluings will take place at the points $0_{2k-1}, \infty_{2k-1}, a_{2k-1,1}$, and $a_{2k-1,2}$. In $\overline{\mathbb{C}}_{2n}$, the gluings will take place at the points $b_{2n-1,1}$ and $b_{2n-1,2}$ and the ends are located at 0_{2n} and ∞_{2n}. We can fix the location of another point on each $\overline{\mathbb{C}}_k$. Thus, fix $a_{2k-1,1} = b_{2k-1,1} = 1$ for $k = 1, 2, \dots, n$. For the purposes of the construction, we label $0_{2k} = a_{2k,1}$, $\infty_{2k+1} = b_{2k,1}$, $\infty_{2k} = a_{2k,2}$ and $0_{2k+1} = b_{2k,2}$.

Let $\epsilon > 0$ be small enough such that the distance between any of the above points is greater than $\epsilon/2$. For each $k = 1, 2, \dots, 2n-1$ and $i = 1, 2$, let $t_{k,i} \in \mathbb{C}$ with $0 < |t_{k,i}| < \epsilon^2$. Let $r_{2k-1,i} = z - a_{2k-1,i}$ be a complex coordinate near $a_{2k-1,i}$ in $\overline{\mathbb{C}}_{2k-1}$, and let $s_{2k-1,i} = z - b_{2k-1,i}$ be a complex coordinate near $b_{2k-1,i}$ in $\overline{\mathbb{C}}_{2k}$. Let $r_{2k,1} = z$ and $r_{2k,2} = 1/z$ be complex coordinates near $a_{2k,1} = 0_{2k}$ and $a_{2k,2} = \infty_{2k}$, respectively, in $\overline{\mathbb{C}}_{2k}$, and let $s_{2k,1} = 1/z$ and $s_{2k,2} = z$ be complex coordinates near $b_{2k,1} = \infty_{2k+1}$ and $b_{2k,2} = 0_{2k+1}$, respectively, in $\overline{\mathbb{C}}_{2k+1}$. Remove the disks $|r_{k,i}| < \dfrac{|t_{k,i}|}{\epsilon}$ and $|s_{k,i}| < \dfrac{|t_{k,i}|}{\epsilon}$ from $\overline{\mathbb{C}}_k$ and $\overline{\mathbb{C}}_{k+1}$, respectively, and identify the points in $\overline{\mathbb{C}}_k$ satisfying

$$\frac{|t_{k,i}|}{\epsilon} < |r_{k,i}| < \epsilon$$

with the points in $\overline{\mathbb{C}}_{k+1}$ satisfying

$$\frac{|t_{k,i}|}{\epsilon} < |s_{k,i}| < \epsilon$$

so that

$$r_{k,i}s_{k,i} = t_{k,i}.$$

Let Σ_t be the Riemann surface created by this gluing procedure, with $t = \{t_{k,i}\}$ and punctures at $0_1, \infty_1, 0_{2n}$, and ∞_{2n}. See Figure 2.1.

If $t_{k,i} = 0$ then then the points $a_{k,i}$ and $b_{k,i}$ are identified and Σ_t has a node corresponding to those identified points. When $t = 0$, we get Σ_0, a singular Riemann surface with nodes, the disjoint union of $2n$ copies of the Riemann sphere with the following pairs of points identified: $(a_{k,i}, b_{k,i})$ for $k = 1, 2, \ldots, 2n-1$ and $i = 1, 2$.

In order to use the implicit function theorem to solve the period problem, it is necessary to set

$$t_{k,i} = e^{-\frac{u_{k,i}+iv_{k,i}}{x^2}}$$

with $u_{k,i} > 0$. Note that $t_{k,i} \to 0$ as $x \to 0$ for $i = 1, 2$ and $k = 1, 2, \ldots, 2n-1$. When the surface is rescaled by $-x^2$, the translation between consecutive copies of Scherk's surface is given by the $v_{k,i}$ terms, as is demonstrated in section 5.

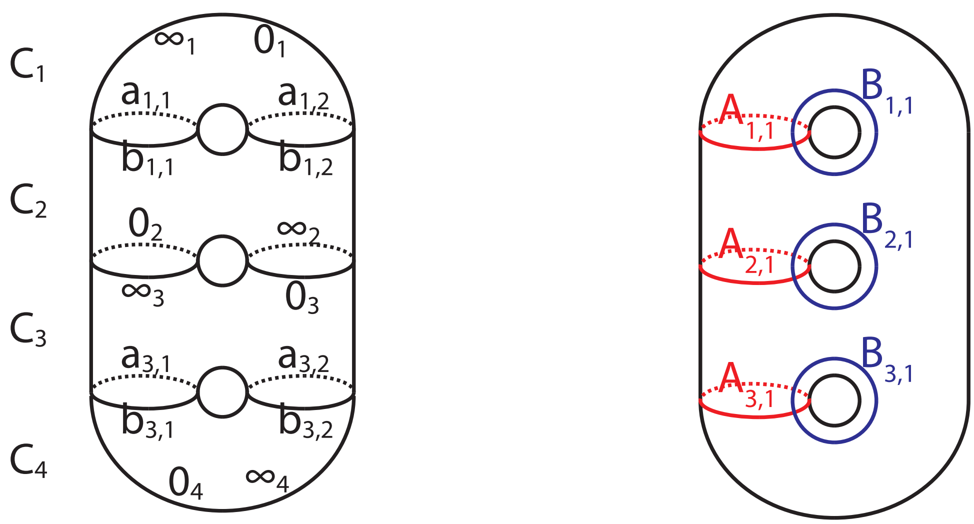

FIGURE 2.1. Models for Σ_t and its homology basis

2.1. **Homology basis of Σ_t.** Let $C(a_{k,i})$ be the circle $|r_{k,i}| = \epsilon$ in $\overline{\mathbb{C}}_k$ and $C(b_{k,i})$ be the circle $|s_{k,i}| = \epsilon$ in $\overline{\mathbb{C}}_{k+1}$. Then $C(b_{k,i})$ is homologous to $-C(a_{k,i})$. Let $A_{k,i}$ be the homology class of the circle $C(a_{k,i})$, positively oriented in $\mathbb{C}_k$. Let $B_{k,1}$ be the cycles in Figure 2.1 that $A_{k,1} \cdot B_{k,1} = 1$, $A_{k,2} \cdot B_{k,1} = -1$. The set $\{A_{k,1}, B_{k,1} : k = 1, \dots, 2n-1\}$ forms a homology basis of Σ_t.

3. Lemmas on differentials

There are four lemmas on differentials on our surfaces Σ_t we need in order to solve the period and conformality problems. The lemmas are the same as in section 5.3 of [13], with slight modifications to the first two because the surfaces we are constructing have ends and Σ_0 consists of a disjoint union of Riemann spheres as opposed to tori. We prove the first two and refer the reader to [13] for the proofs of the third and fourth lemmas, which are exactly the same.

Definition 3.1. ([7]) A regular q-differential ω on Σ is a form of type $(q, 0)$ which

(1) ω has poles at the punctures of Σ with orders at most q and
(2) if a and b are two punctures that are paired to form a node then $\operatorname{Res}_b \omega = (-1)^q \operatorname{Res}_a \omega$.

Let $\Omega^1(\Sigma_t)$ be the space of regular 1-differentials on Σ_t with at most single-order poles at the ends. Let $\Omega^2(\Sigma_t)$ be the space of regular 2-differentials on Σ_t with at most double-order poles at the ends. The dimension of $\Omega_1(\Sigma_t)$ is $g + e - 1$, and the dimension of $\Omega^2(\Sigma_t)$ is $3g - 3 + 2e$, where g is the genus and e is the number of punctures.

Propositions 4.1 and 5.1 of Masur [7] state that there exist a basis of $\Omega^q(\Sigma_t)$ for $q = 1, 2$ that depends holomorphically on t in a neighborhood of $t = 0$. That is, for any $\delta > 0$, the restriction of $\omega_t \in \Omega^q(\Sigma_t)$ to the domain

$$\Omega_\delta = \{z \in \Sigma_t : |v_{k,i}| > \delta, |w_{k,i}| > \delta, \forall k, i\}$$

depends holomorphically on $z \in \Omega_\delta$ and t in a neighborhood of 0.

Lemma 3.2. *For t close to 0, the map*

$$\omega \mapsto \left(\int_{C(0_1)} \omega, \int_{C(\infty_1)} \omega, \underbrace{\int_{C(0_{2k})} \omega, \int_{C(a_{2k-1,1})} \omega}_{k \in \{1, \dots, n\}} \right)$$

is an isomorphism from $\Omega^1(\Sigma_t)$ to $\mathbb{C}^{g+e-1} = \mathbb{C}^{2n+2}$.

Proof. We prove this for $t = 0$ by showing that the map is injective and conclude by continuity that it holds for small t. Let ω be in the kernel of this map. Then ω is holomorphic at $0_1, \infty_1, 0_{2k}, a_{2k-1,1}$ for $k = 1, 2, \ldots, n$. By the definition of a regular differential, ω is also holomorphic at ∞_{2k+1} for $k = 1, 2, \ldots, n-1$ and $b_{2k-1,1}$ for $k = 1, 2, \ldots, n$. By the residue theorem on $\overline{\mathbb{C}}_1$, ω is holomorphic at $a_{1,2}$. By the definition of a regular quadratic differential, ω holomorphic at $a_{1,2}$ implies that ω is holomorphic at $b_{1,2}$. By the residue theorem on $\overline{\mathbb{C}}_2$, ω is holomorphic at ∞_2. Iterating this procedure for each successive k with $2 \leq k \leq n$, we get that ω is holomorphic at $b_{2k-1,2}, \infty_{2k}$ for $k = 1, 2, \ldots, n$. Hence, ω is holomorphic on each sphere, completing the proof. $\square$

Lemma 3.3. *For t close to 0, the map*

$$L(\psi) = \left(\underbrace{\int_{C(0_k)} \frac{z\psi}{dz}}_{k\in\{1,2n\}}, \underbrace{\int_{C(a_{2k-1,i})} \frac{(z-a_{2k-1,i})\psi}{dz}, \int_{C(a_{2k-1,i})} \frac{\psi}{dz}}_{k\in\{1,\ldots,n\}, i\in\{1,2\}}, \underbrace{\int_{C(0_{2k})} \frac{\psi}{dz}, \int_{C(b_{2k-1,2})} \frac{\psi}{dz}}_{k\in\{1,\ldots,n\}} \right)$$

is an isomorphism from $\Omega^2(\Sigma_t)$ to $\mathbb{C}^{3g+5} = \mathbb{C}^{6n+2}$.

Proof. We prove this for $t = 0$ by showing that the map is injective and conclude by continuity that it holds for small t. Let ψ be in the kernel of L. We note that since dz has a double-order pole at each ∞_j, $\frac{\psi}{dz}$ is holomorphic at ∞_j for $j = 1, 2, \ldots, 2n$. Combined with the first two components of $L(\psi)$ equaling zero, this implies that $\frac{\psi}{dz}$ has at most simple poles at $a_{2k-1,i}, b_{2k-1,i}$, and 0_{2k} for $k = 1, 2, \ldots, n$ and $i = 1, 2$. The third, fourth, and fifth components of $L(\psi)$ all equaling zero imply that $\frac{\psi}{dz}$ is holomorphic at $a_{2k-1,i}, b_{2k-1,2}, 0_{2k}$ for $k = 1, 2, \ldots, n$ and $i = 1, 2$. By the residue theorem on each $\overline{\mathbb{C}}_j$, $\frac{\psi}{dz}$ is holomorphic at $0_{2k-1}, b_{2k-1,1}$ for $k = 1, 2, \ldots, n$, and so $\frac{\psi}{dz}$ is holomorphic on each sphere $\overline{\mathbb{C}}_j$ for $j = 1, 2, \ldots, 2n$. Hence, $\frac{\psi}{dz} = 0$ on each sphere, completing the proof. $\square$

Lemma 3.4. *Let $\omega_t \in \Omega^1(\Sigma_t)$ be defined by prescribing the periods as in Lemma 3.2. Then on a compact subset of Σ_t, $\partial\omega_t/\partial t_{k,i}$ at $t = 0$ is a meromorphic 1-form with double-order poles at $a_{k,i}$ and $b_{k,i}$, is holomorphic otherwise, and has vanishing periods at all of the cycles in Lemma 3.2. Furthermore, the principal part of the poles are*

$$\frac{dz}{(z-a_{k,i})^2} \frac{-1}{2\pi i} \int_{C(b_{k,i})} \frac{\omega_0}{z-b_{k,i}} \ \text{at } a_{k,i}$$
$$\frac{dz}{(z-b_{k,i})^2} \frac{-1}{2\pi i} \int_{C(a_{k,i})} \frac{\omega_0}{z-a_{k,i}} \ \text{at } b_{k,i}.$$

Note that for $a_{2k,1} = 0_{2k}$ and $b_{2k,1} = \infty_{2k+1}$, the principal part of the pole at 0_{2k} is

$$\frac{dz}{z^2}\frac{-1}{2\pi i}\int_{C(\infty_{2k+1})} z\omega_0.$$

Lemma 3.5. *The difference*

$$\int_{a_{k,i}+\epsilon}^{b_{k,i}+\epsilon} \omega_t - \frac{\log t_{k,i}}{2\pi i}\int_{C(a_{k,i})} \omega_t$$

is a well defined analytic function of $t_{k,i}$ which extends analytically to $t_{k,i} = 0$.

4. Weierstrass data ϕ_1, ϕ_2, ϕ_3

4.1. **Definition of ϕ_1, ϕ_2, ϕ_3.** After rotating and rescaling, if necessary, let the lattice Λ of the translation vectors be generated by the vectors $(0, 2\pi, 0)$ and $(2\pi T_1, 2\pi T_2, 0)$. We need to define meromorphic one-forms on Σ_t such that

$$\begin{aligned}
\operatorname{Re}\int_{A_{2k-1,1}} (\phi_1, \phi_2, \phi_3) &= (0, 2\pi, 0)\\
\operatorname{Re}\int_{A_{2k,1}} (\phi_1, \phi_2, \phi_3) &= (2\pi T_1, 2\pi T_2, 0)\\
\operatorname{Re}\int_{B_{k,1}} (\phi_1, \phi_2, \phi_3) &= (0, 0, 0)\\
\operatorname{Re}\int_{C(0_1)} (\phi_1, \phi_2, \phi_3) &= (2\pi T_1, 2\pi T_2, 0)\\
\operatorname{Re}\int_{C(\infty_1)} (\phi_1, \phi_2, \phi_3) &= -(2\pi T_1, 2\pi T_2, 0)
\end{aligned}$$

Definition 4.1. Define ϕ_1, ϕ_2, ϕ_3 by prescribing their residues along the cycles $C(a_{2k-1,1})$, $C(0_{2k})$, for $k = 1, \dots, n$, $C(0_1)$, and $C(\infty_1)$ as follows.

- $\int_{C(a_{2k-1,1})} (\phi_1, \phi_2, \phi_3) = 2\pi i\,(\alpha_{2k-1,1,1}, \alpha_{2k-1,1,2} - i, \alpha_{2k-1,1,3})$
- $\int_{C(0_{2k})} (\phi_1, \phi_2, \phi_3) = 2\pi i\,(\gamma_{2k,0,1} - iT_1, \gamma_{2k,0,2} - iT_2, \gamma_{2k,0,3})$
- $\int_{C(0_1)} (\phi_1, \phi_2, \phi_3) = 2\pi i\,(\gamma_{1,0,1} - iT_1, \gamma_{1,0,2} - iT_2, \gamma_{1,0,3})$
- $\int_{C(\infty_1)} (\phi_1, \phi_2, \phi_3) = 2\pi i\,(\gamma_{1,\infty,1} + iT_1, \gamma_{1,\infty,2} + iT_2, \gamma_{1,\infty,3})$

4.2. **Parameters count.** There are $4n$ real parameters from the location of the punctures $a_{k,2}, b_{k,2}$. The parameters

$$t_{k,i} = e^{-\frac{u_{k,i}+iv_{k,i}}{x^2}}$$

give us $8n-3$ real parameters. However, the map

$$(x, u_{k,i} + iv_{k,i}) \mapsto \big(\lambda x, \lambda^2(u_{k,i} + iv_{k,i})\big)$$

yields the same $t_{k,i}$, leaving $8n-4$ real parameters. There are $6n+8$ real period parameters $\alpha_{k,i,j}, \gamma_{k,i,j}$, and T_j. Thus, there are a total of $18n+4$ real parameters.

There are $6n+2$ complex equations from $\phi_1^2 + \phi_2^2 + \phi_3^2$ and $6n-3$ real equations from the period problem. So, there are a total of $18n+1$ real equations, leaving three extra parameters. One of the extra parameters will be used to ensure the surfaces are embedded.

4.3. **Weierstrass data on Σ_0.** When $x = 0$, $t_{k,i} = 0$ for $k = 1, 2, \ldots, 2n-1$ and $i = 1, 2$, and the Weierstrass data splits into Weierstrass data on each Riemann sphere $\overline{\mathbb{C}}_k$. Define $\gamma_{0,0,j} = -\gamma_{1,\infty,j}$.

On $\overline{\mathbb{C}}_{2k-1}$ for $k \in \{1, 2, \ldots, n\}$, ϕ_1, ϕ_2, and ϕ_3 are given respectively by

$$\left(\frac{\alpha_{2k-1,1,1}}{z-1} + \frac{-\alpha_{2k-1,1,1} - \gamma_{1,0,1} - \gamma_{1,\infty,1}}{z - a_{2k-1,2}} + \frac{\gamma_{2k-2,0,1} + \gamma_{1,0,1} + \gamma_{1,\infty,1} - iT_1}{z}\right) dz,$$

$$\left(\frac{\alpha_{2k-1,1,2} - i}{z-1} + \frac{-\alpha_{2k-1,1,2} - \gamma_{1,0,2} - \gamma_{1,\infty,2} + i}{z - a_{2k-1,2}} + \frac{\gamma_{2k-2,0,2} + \gamma_{1,0,2} + \gamma_{1,\infty,2} - iT_2}{z}\right) dz,$$

$$\left(\frac{\alpha_{2k-1,1,3}}{z-1} + \frac{-\alpha_{2k-1,1,3} - \gamma_{1,0,3} - \gamma_{1,\infty,3}}{z - a_{2k-1,2}} + \frac{\gamma_{2k-2,0,3} + \gamma_{1,0,3} + \gamma_{1,\infty,3}}{z}\right) dz.$$

On $\overline{\mathbb{C}}_{2k}$ for $k \in \{1, 2, \ldots, n\}$, ϕ_1, ϕ_2, and ϕ_3 are given respectively by

$$\left(-\frac{\alpha_{2k-1,1,1}}{z-1} + \frac{\alpha_{2k-1,1,1} + \gamma_{1,0,1} + \gamma_{1,\infty,1}}{z - b_{2k-1,2}} + \frac{\gamma_{2k,0,1} - iT_1}{z}\right) dz,$$

$$\left(\frac{-\alpha_{2k-1,1,2} + i}{z-1} + \frac{\alpha_{2k-1,1,2} + \gamma_{1,0,2} + \gamma_{1,\infty,2} - i}{z - b_{2k-1,2}} + \frac{\gamma_{2k,0,2} - iT_2}{z}\right) dz,$$

$$\left(-\frac{\alpha_{2k-1,1,3}}{z-1} + \frac{\alpha_{2k-1,1,3} + \gamma_{1,0,3} + \gamma_{1,\infty,3}}{z - b_{2k-1,2}} + \frac{\gamma_{2k,0,3}}{z}\right) dz.$$

From this point forward, when $x = 0$ set

- $a^0_{2k-1,2} = \frac{T_2^0 - 1}{T_2^0 + 1}$; $b^0_{2k-1,2} = \frac{T_2^0+1}{T_2^0-1}$; $T_1^0 = \sqrt{1 - (T_2^0)^2}$;
- $\alpha^0_{2k-1,1,1} = \alpha^0_{2k-1,1,2} = 0$; $\alpha^0_{2k-1,1,3} = 1$;
- $\gamma^0_{2k,0,1} = \gamma^0_{2k,0,2} = 0$; $\gamma^0_{2k,0,3} = 1$;

- $\gamma^0_{1,0,1} = \gamma^0_{1,\infty,1} = \gamma^0_{1,0,2} = \gamma^0_{1,\infty,2} = 0; \gamma^0_{1,0,3} = \gamma^0_{1,\infty,3} = -1;$

for $k = 1, 2, \ldots, n$. Then the Weierstrass data is a bit more manageable:

$$\Phi_{2k-1} = \left(-\frac{i\sqrt{1-T_2^2}}{z}, \frac{-i}{z-1} + \frac{i}{z - \frac{T_2-1}{T_2+1}} - \frac{iT_2}{z}, \frac{1}{z-1} + \frac{1}{z - \frac{T_2-1}{T_2+1}} - \frac{1}{z} \right) dz$$

$$\Phi_{2k} = \left(-\frac{i\sqrt{1-T_2^2}}{z}, \frac{i}{z-1} - \frac{i}{z - \frac{T_2+1}{T_2-1}} - \frac{iT_2}{z}, -\frac{1}{z-1} - \frac{1}{z - \frac{T_2+1}{T_2-1}} + \frac{1}{z} \right) dz$$

for $k = 1, 2, \ldots, n$.

In this case, $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$ and so the conformal equations are solved when $x = 0$, and (ϕ_1, ϕ_2, ϕ_3) is the Weierstrass representation of the desired doubly periodic Scherk surfaces, with angle $\theta = \cos^{-1}(T_2)$ between the top and bottom ends.

5. Period Problem

The periods yet to be solved are

$$\mathrm{Re} \int_{B_{k,1}} \phi_j$$

for $k = 1, 2, \ldots, 2n-1$ and $j = 1, 2, 3$. In order to compute these periods, we choose representatives of $B_{2k-1,1}$ for $k = 1, 2, \ldots, n$, defined as the concatenation of the following paths:

(1) A path from $r_{2k-1,1} = \epsilon$ to $s_{2k-1,1} = \epsilon$, going through the corresponding neck;
(2) A path from $b_{2k-1,1} + \epsilon$ to $b_{2k-1,2} + \epsilon$ in $\mathbb{C}_{2k}$, avoiding punctures;
(3) A path from $s_{2k-1,2} = \epsilon$ to $r_{2k-1,2} = \epsilon$, going through the corresponding neck;
(4) A path from $a_{2k-1,2} + \epsilon$ to $a_{2k-1,1} + \epsilon$ in $\mathbb{C}_{2k-1}$, avoiding punctures.

Also, choose representatives of $B_{2k,1}$ for $k = 1, 2, \ldots, n$, defined as concatenation of the following paths:

(1) A path from $r_{2k,1} = \epsilon$ to $s_{2k,1} = \epsilon$, going through the corresponding neck;
(2) A path from $1/\epsilon$ to ϵ in $\mathbb{C}_{2k+1}$, avoiding punctures;
(3) A path from $s_{2k,2} = \epsilon$ to $r_{2k,2} = \epsilon$, going through the corresponding neck;
(4) A path from $1/\epsilon$ to ϵ in $\mathbb{C}_{2k}$, avoiding punctures.

Note that each ϕ_j is analytic with respect to x on paths 2 and 4. Using Lemma 3.5 and

$$t_{k,i} = e^{-\frac{u_{k,i} + iv_{k,i}}{x^2}},$$

$$\begin{aligned}\operatorname{Re}\int_{B_{2k-1,1}}\phi_j &= \operatorname{Re}\left[\frac{\log t_{2k-1,1}}{2\pi i}\int_{C(a_{2k-1,1})}\phi_j - \frac{\log t_{2k-1,2}}{2\pi i}\int_{C(a_{2k-1,2})}\phi_j\right] + \text{analytic}\\ &= -\frac{1}{x^2}\left[u_{2k-1,1}\alpha_{2k-1,1,j} + u_{2k-1,2}(\alpha_{2k-1,1,j}+\gamma_{2k-1,0,j}+\gamma_{2k-1,\infty,j})\right.\\ &\qquad \left. +(v_{2k-1,1}+v_{2k-1,2})\delta_{2,j}\right] + \text{analytic}\end{aligned}$$

and

$$\begin{aligned}\operatorname{Re}\int_{B_{2k,1}}\phi_j &= \operatorname{Re}\left[\frac{\log t_{2k,1}}{2\pi i}\int_{C(0_{2k})}\phi_j + \frac{\log t_{2k,2}}{2\pi i}\int_{C(0_{2k+1})}\phi_j\right] + \text{analytic}\\ &= -\frac{1}{x^2}\left[u_{2k,1}\gamma_{2k,0,j} + u_{2k,2}(\gamma_{2k,0,j}+\gamma_{1,0,j}+\gamma_{1,\infty,j}) + v_{2k,1}T_j(1-\delta_{1,j})\right.\\ &\qquad \left. +v_{2k,2}T_j(1-\delta_{1,j})\right] + \text{analytic}\end{aligned}$$

The expressions $-x^2\operatorname{Re}\int_{B_{k,1}}\phi_j$ extend smoothly to $x=0$. From the $B_{2k-1,1}$ and B_{2k} periods, we get the equations

$$\begin{aligned}u_{2k-1,1}\alpha_{2k-1,1,1} + u_{2k-1,2}(\alpha_{2k-1,1,1}+\gamma_{1,0,1}+\gamma_{1,\infty,1}) &= 0\\ u_{2k-1,1}\alpha_{2k-1,1,2} + u_{2k-1,2}(\alpha_{2k-1,1,2}+\gamma_{1,0,2}+\gamma_{1,\infty,2}) + v_{2k-1,1}+v_{2k-1,2} &= 0\\ u_{2k-1,1}\alpha_{2k-1,1,3} + u_{2k-1,2}(\alpha_{2k-1,1,3}+\gamma_{1,0,3}+\gamma_{1,\infty,3}) &= 0\\ u_{2k,1}\gamma_{2k,0,1} + u_{2k,2}(\gamma_{2k,0,1}+\gamma_{1,0,1}+\gamma_{1,\infty,1}) + (v_{2k,1}+v_{2k,2})T_1 &= 0\\ u_{2k,1}\gamma_{2k,0,2} + u_{2k,2}(\gamma_{2k,0,2}+\gamma_{1,0,2}+\gamma_{1,\infty,2}) + (v_{2k,1}+v_{2k,2})T_2 &= 0\\ u_{2k,1}\gamma_{2k,0,3} + u_{2k,2}(\gamma_{2k,0,3}+\gamma_{1,0,3}+\gamma_{1,\infty,3}) &= 0\end{aligned}$$

which, when we plug in the values for the variables from section 4.3, reduce to

$$\begin{aligned}0 &= 0\\ v_{2k-1,1}+v_{2k-1,2} &= 0\\ u_{2k-1,1}-u_{2k-1,2} &= 0\\ (v_{2k,1}+v_{2k,2})T_1 &= 0\\ (v_{2k,1}+v_{2k,2})T_2 &= 0\\ u_{2k,1}-u_{2k,2} &= 0\end{aligned}$$

with solution $u_{2k-1,2}=u_{2k-1,1}, v_{2k-1,2}=-v_{2k-1,1}, u_{2k,2}=u_{2k,1}, v_{2k,2}=-v_{2k,1}$.

We solve the period problem by using the implicit function theorem to find solutions to the equations

$$-x^2\operatorname{Re}\int_{B_{k,1}}\phi_j = 0$$

for x close to 0.

Proposition 5.1. *For x close to 0, there exist unique $(\alpha_{1,1,1}, \gamma_{1,0,2}, \gamma_{1,0,3}, \gamma_{2,0,1}, v_{2,2}, u_{2,2})$ in a neighborhood of $\left(0, 0, -1, 0, v^0_{2,2}, u^0_{2,2}\right)$, depending on the other parameters, such that*

$$\int_{C(B_{k,1})} \phi_j = 0$$

for $k = 1, 2$ and $j = 1, 2, 3$.

Proof. When $x = 0$, the partial differential of the period equations for ϕ_1, ϕ_2, ϕ_3 on $B_{1,1}$ and $B_{2,1}$ is

$$\begin{bmatrix} 2u_{1,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & u_{1,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & u_{1,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2u_{2,1} & T_1 & 0 \\ 0 & u_{2,2} & 0 & 0 & T_2 & 0 \\ 0 & 0 & u_{2,1} & 0 & 0 & -1 \end{bmatrix}$$

giving a matrix with full rank, assuming $u_{1,1}, u_{1,2}, u_{2,1}$, and T_2 are all nonzero. The result follows from the implicit function theorem. $\square$

Proposition 5.2. *For x close to 0, there exist unique $(\alpha_{2k-1,1,1}, v_{2k-1,2}, u_{2k-1,2}, \gamma_{2k,0,1}, v_{2k,2}, u_{2k,2})$, $k > 1$, in a neighborhood of $\left(0, v^0_{2k-1,2}, u^0_{2k-1,2}, 0, v^0_{2k,2}, u^0_{2k,2}\right)$, $k > 1$, depending on the other parameters, such that*

$$\int_{C(B_{\ell,1})} \phi_j = 0$$

for $\ell = 3, \dots, 2n-1$ and $j = 1, 2, 3$.

Proof. The partial differential of the period equations for ϕ_1, ϕ_2, ϕ_3 on $B_{3,1}, \dots, B_{2n-1,1}$ evaluated when $x = 0$ with respect to the variables $(\alpha_{2k-1,1,1}, v_{2k-1,2}, u_{2k-1,2})$ and $(\gamma_{2k,0,1}, v_{2k,2}, u_{2k,2})$ for $k = 2, \dots, n-1$, and $(\alpha_{2n-1,1,1}, v_{2n-1,2}, u_{2n-1,2})$ has 6×6 diagonal blocks of the form

$$\begin{bmatrix} 2u_{2k-1,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2u_{2k,1} & T_1 & 0 \\ 0 & 0 & 0 & 0 & T_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

and a 3×3 diagonal block of the form

$$\begin{bmatrix} 2u_{2n-1,1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

with zeros in all other entries. This matrix has full rank, assuming $u_{2k-1,1} \neq 0$, $u_{2k,1} \neq 0$, $T_2 \neq 0$. The result follows from the implicit function theorem. $\square$

The requirement here that $T_2 \neq 0$ is the part of the construction that rules out having orthogonal period vectors at the limit when we are gluing more than two Scherk surfaces together.

The translation between consecutive copies of Scherk's surface with domains $\overline{\mathbb{C}}_{2k-1}$ and $\overline{\mathbb{C}}_{2k}$ is given by

$$\begin{aligned} -x^2 \operatorname{Re} \int_{\epsilon \in \overline{\mathbb{C}}_{2k}}^{1/\epsilon \in \overline{\mathbb{C}}_{2k+1}} \omega_2 &= -x^2 \left(\operatorname{Re} \int_{\epsilon}^{a_{2k-1,1}+\epsilon} \omega_2 + \operatorname{Re} \int_{a_{2k-1,1}+\epsilon}^{a_{2k,1}+\epsilon} \omega_2 + \operatorname{Re} \int_{b_{2k-1,1}+\epsilon}^{1/\epsilon} \omega_2 \right) \\ &= u_{2k-1,1}\alpha_{2k-1,1,2} + v_{2k-1,1} + o(x^2) \\ &\to v_{2k-1,1} \end{aligned}$$

as $x \to 0$. Similarly, the translation between consecutive copies of Scherk's surface with domains $\overline{\mathbb{C}}_{2k}$ and $\overline{\mathbb{C}}_{2k+1}$ is given by

$$\begin{aligned} -x^2 \left(\operatorname{Re} \int_{b_{2k-1,1}+\epsilon}^{a_{2k+1,1}+\epsilon} \omega_1 + i \operatorname{Re} \int_{b_{2k-1,1}+\epsilon}^{a_{2k+1,1}+\epsilon} \omega_2 \right) &= u_{2k,1}(\gamma_{2k,0,1} + i\gamma_{2k,0,2}) + v_{2k,1}(T_1 + iT_2) \\ &\quad + o(x^2) \\ &\to v_{2k,1}(T_1 + iT_2) \end{aligned}$$

as $x \to 0$.

6. Conformality

The conformal equation

$$Q = \phi_1^2 + \phi_2^2 + \phi_3^2 = 0$$

is solved in this section using Lemma 3.3. In the same manner as how the period problem was solved, we solve the conformal equations from Lemma 3.3 when $x = 0$ and use the implicit function theorem to show they are solved for small x. In fact, when we plug in the values for the variables given in section 4.3, the conformal equations are automatically solved when $x = 0$. In each of the proofs, it is necessary that $T_2 \neq \pm 1$. This is also necessary in order to have linearly independent period vectors.

Proposition 6.1. *For x close to 0, there exists unique $(\gamma_{1,\infty,1}, \gamma_{1,\infty,3}, \gamma_{1,0,1}, T_1)$ in a neighborhood of $\left(0, -1, 0, \sqrt{1 - (T_2^0)^2}\right)$, depending on the other parameters, such that*

$$\int_{C(a_{1,2})} \frac{Q}{dz} = \int_{C(0_1)} z\frac{Q}{dz} = 0.$$

Proof. On $\overline{C}_1$,

$$\mathcal{Q} = \Bigg[\left(\frac{\alpha_{1,1,1}}{z-1} - \frac{\alpha_{1,1,1}+\gamma_{1,0,1}+\gamma_{1,\infty,1}}{z-a_{1,2}} + \frac{\gamma_{1,0,1}-iT_1}{z}\right)^2 + \left(\frac{\alpha_{1,1,2}-i}{z-1} - \frac{\alpha_{1,1,2}+\gamma_{1,0,2}+\gamma_{1,\infty,2}-i}{z-a_{1,2}} + \frac{\gamma_{1,0,2}-iT_2}{z}\right)^2 + \left(\frac{\alpha_{1,1,3}}{z-1} - \frac{\alpha_{1,1,3}+\gamma_{1,0,3}+\gamma_{1,\infty,3}}{z-a_{1,2}} + \frac{\gamma_{1,0,3}}{z}\right)^2\Bigg] dz^2 + o(x).$$

Then,

$$\int_{C(a_{1,2})} \frac{\mathcal{Q}}{dz} = 4\pi i \Bigg[(-\alpha_{1,1,1}-\gamma_{1,0,1}-\gamma_{1,\infty,1})\left(\frac{\alpha_{1,1,1}}{a_{1,2}-1} + \frac{\gamma_{1,0,1}-iT_1}{a_{1,2}}\right) + (-\alpha_{1,1,2}-\gamma_{1,0,2}-\gamma_{1,\infty,2}+i)\left(\frac{\alpha_{1,1,2}-i}{a_{1,2}-1} + \frac{\gamma_{1,0,2}-iT_2}{a_{1,2}}\right) + (-\alpha_{1,1,3}-\gamma_{1,0,3}-\gamma_{1,\infty,3})\left(\frac{\alpha_{1,1,3}}{a_{1,2}-1} + \frac{\gamma_{1,0,3}}{a_{1,2}}\right)\Bigg] + o(x)$$

and

$$\int_{C(0_1)} z\frac{\mathcal{Q}}{dz} = 2\pi i\left((\gamma_{1,0,1}-iT_1)^2 + (\gamma_{1,0,2}-iT_2)^2 + \gamma_{1,0,3}^2\right) + o(x).$$

When $x=0$, the nonzero derivatives are

$$\frac{\partial}{\partial\gamma_{1,\infty,1}} \int_{C(a_{1,2})} \frac{\mathcal{Q}}{dz} = 4\pi i\left[\frac{iT_1}{a_{1,2}}\right] = -\frac{4\pi\sqrt{1-T_2^2}(T_2+1)}{T_2-1}$$

$$\frac{\partial}{\partial\gamma_{1,\infty,3}} \int_{C(a_{1,2})} \frac{\mathcal{Q}}{dz} = 4\pi i\left[-\frac{1}{a_{1,2}-1} + \frac{1}{a_{1,2}}\right] = \frac{2\pi i(T_2+1)^2}{T_2-1}$$

$$\frac{\partial}{\partial\gamma_{1,0,1}} \int_{C(a_{1,2})} \frac{\mathcal{Q}}{dz} = 4\pi i\left[\frac{iT_1}{a_{1,2}}\right] = -\frac{4\pi\sqrt{1-T_2^2}(T_2+1)}{T_2-1}$$

$$\frac{\partial}{\partial\gamma_{1,0,1}} \int_{C(0_1)} z\frac{\mathcal{Q}}{dz} = 4\pi\sqrt{1-T_2^2}$$

$$\frac{\partial}{\partial T_1} \int_{C(0_1)} z\frac{\mathcal{Q}}{dz} = -4\pi i\sqrt{1-T_2^2}.$$

The matrix of partial derivatives is invertible, and the statement follows from the implicit function theorem. □

Proposition 6.2. *For x close to 0, there exists unique $b_{2k-1,2}$ in a neighborhood of $\dfrac{T_2^0+1}{T_2^0-1}$, depending on the other parameters, such that*

$$\int_{C(b_{2k-1,2})} \frac{\mathcal{Q}}{dz} = 0$$

for $k = 1, 2, \ldots, n$.

Proof. On $\overline{C}_{2k}$,

$$\begin{aligned}\mathcal{Q} = \Bigg[&\left(-\frac{\alpha_{2k-1,1,1}}{z-1} + \frac{\alpha_{2k-1,1,1}+\gamma_{1,0,1}+\gamma_{1,\infty,1}}{z-b_{2k-1,2}} + \frac{\gamma_{2k,0,1}-iT_1}{z}\right)^2 + \\ &\left(\frac{-\alpha_{2k-1,1,2}+i}{z-1} + \frac{\alpha_{2k-1,1,2}+\gamma_{1,0,2}+\gamma_{1,\infty,2}-i}{z-b_{2k-1,2}} + \frac{\gamma_{2k,0,2}-iT_2}{z}\right)^2 + \\ &\left(-\frac{\alpha_{2k-1,1,3}}{z-1} + \frac{\alpha_{2k-1,1,3}+\gamma_{1,0,3}+\gamma_{1,\infty,3}}{z-b_{2k-1,2}} + \frac{\gamma_{2k,0,3}}{z}\right)^2\Bigg] dz^2 + o(x).\end{aligned}$$

Then,

$$\begin{aligned}\int_{C(b_{2k-1,2})} \frac{\mathcal{Q}}{dz} = 4\pi i \Bigg[&(\alpha_{2k-1,1,1}+\gamma_{1,0,1}+\gamma_{1,\infty,1})\left(-\frac{\alpha_{2k-1,1,1}}{b_{2k-1,2}-1} + \frac{\gamma_{2k,0,1}-iT_1}{b_{2k-1,2}}\right) + \\ &(\alpha_{2k-1,1,2}+\gamma_{1,0,2}+\gamma_{1,\infty,2}-i)\left(\frac{-\alpha_{2k-1,1,2}+i}{b_{2k-1,2}-1} + \frac{\gamma_{2k,0,2}-iT_2}{b_{2k-1,2}}\right) + \\ &(\alpha_{2k-1,1,3}+\gamma_{1,0,3}+\gamma_{1,\infty.3})\left(-\frac{\alpha_{2k-1,1,3}}{b_{2k-1,2}-1} + \frac{\gamma_{2k,0,3}}{b_{2k-1,2}}\right)\Bigg] + o(x).\end{aligned}$$

When $x = 0$, the nonzero derivatives are

$$\frac{\partial}{\partial b_{2k-1,2}} \int_{C(b_{2k-1,2})} \frac{\mathcal{Q}}{dz} = 4\pi i \left[\frac{-2}{(b_{2k-1,2}-1)^2} + \frac{T_2+1}{b_{2k-1,2}^2}\right] = -\frac{2\pi i (T_2-1)^3}{T_2+1}.$$

The matrix of partial derivatives is invertible, and the statement follows from the implicit function theorem. □

Proposition 6.3. *For x close to 0, there exists unique $t_{2k,1}$ in a neighborhood of $t_{2k,1}^0$, depending on the other parameters, such that*

$$\int_{C(0_{2k})} \frac{\mathcal{Q}}{dz} = 0$$

for $k = 1, 2, \ldots, n-1$.

Proof. By Lemma 3.4, when $x = 0$, the principal part of $\frac{\partial \phi_1}{\partial t_{2k,1}}$ at 0_{2k} is

$$\frac{-dz}{2\pi i z^2} \int_{C(\infty_{2k+1})} z \left(-\frac{iT_1}{z} \right) dz = 0,$$

the principal part of $\frac{\partial \phi_2}{\partial t_{2k,1}}$ at 0_{2k} is

$$\frac{-dz}{2\pi i z^2} \int_{C(\infty_{2k+1})} z \left(\frac{-i}{z-1} + \frac{i}{z - a_{2k+1,2}} - \frac{iT_2}{z} \right) dz = -\frac{i(1 - a_{2k+1,2})}{z^2} dz = -\frac{2idz}{(T_2+1)z^2},$$

and the principal part of $\frac{\partial \phi_3}{\partial t_{2k,1}}$ at 0_{2k} is

$$\frac{-dz}{2\pi i z^2} \int_{C(\infty_{2k+1})} z \left(\frac{1}{z-1} + \frac{1}{z - a_{2k+1,2}} - \frac{1}{z} \right) dz = \frac{1 + a_{2k+1,2}}{z^2} dz = \frac{2T_2 dz}{(T_2+1)z^2}.$$

Using the expression for $\mathcal{Q}$ on $\overline{C}_{2k}$ from the proof of Proposition 6.2 when $x = 0$,

$$\begin{aligned}
\frac{\partial}{\partial t_{2k,1}} \int_{C(0_{2k})} \frac{\mathcal{Q}}{dz} &= \int_{C(0_{2k})} \frac{2\phi_1}{dz} \frac{\partial \phi_1}{\partial t_{2k,1}} + \frac{2\phi_2}{dz} \frac{\partial \phi_2}{\partial t_{2k,1}} + \frac{2\phi_3}{dz} \frac{\partial \phi_3}{\partial t_{2k,1}} \\
&= 2 \int_{C(0_{2k})} \left[-\frac{2i}{(T_2+1)z^2} \left(\frac{i}{z-1} - \frac{i}{z - b_{2k-1,2}} - \frac{iT_2}{z} \right) + \right. \\
&\qquad \left. \frac{2T_2}{(T_2+1)z^2} \left(-\frac{1}{z-1} - \frac{1}{z - b_{2k-1,2}} + \frac{1}{z} \right) \right] dz \\
&= 4\pi i \left[-\frac{2}{T_2+1} + \frac{2(T_2-1)^2}{(T_2+1)^3} + \frac{2T_2}{T_2+1} + \frac{2T_2(T_2-1)^2}{(T_2+1)^3} \right] \\
&= \frac{16\pi i T_2(T_2-1)}{(T_2+1)^2}.
\end{aligned}$$

The matrix of partial derivatives is invertible, and the statement follows from the implicit function theorem. $\square$

Proposition 6.4. *For x close to 0, there exists unique $a_{2k-1,2}$ in a neighborhood of $\dfrac{T_2^0 - 1}{T_2^0 + 1}$, depending on the other parameters, such that*

$$\int_{C(a_{2k-1,1})} \frac{\mathcal{Q}}{dz} = 0$$

for $k = 1, 2, \ldots, n$.

Proof. On $\overline{C}_{2k-1}$,

$$\mathcal{Q} = \left[\left(\frac{\alpha_{2k-1,1,1}}{z-1} + \frac{-\alpha_{2k-1,1,1} - \gamma_{1,0,1} - \gamma_{1,\infty,1}}{z - a_{2k-1,2}} + \frac{\gamma_{2k-2,0,1} + \gamma_{1,0,1} + \gamma_{1,\infty,1} - iT_1}{z}\right)^2 + \right.$$
$$\left(\frac{\alpha_{2k-1,1,2} - i}{z-1} + \frac{-\alpha_{2k-1,1,2} - \gamma_{1,0,2} - \gamma_{1,\infty,2} + i}{z - a_{2k-1,2}} + \frac{\gamma_{2k-2,0,2} + \gamma_{1,0,2} + \gamma_{1,\infty,2} - iT_2}{z}\right)^2 +$$
$$\left.\left(\frac{\alpha_{2k-1,1,3}}{z-1} + \frac{-\alpha_{2k-1,1,3} - \gamma_{1,0,3} - \gamma_{1,\infty,3}}{z - a_{2k-1,2}} + \frac{\gamma_{2k-2,0,3} + \gamma_{1,0,3} + \gamma_{1,\infty,3}}{z}\right)^2\right] dz^2 + o(x).$$

Then,

$$\int_{C(a_{2k-1,1})} \frac{\mathcal{Q}}{dz}$$
$$=4\pi i \left[\alpha_{2k-1,1,1}\left(\frac{-\alpha_{2k-1,1,1} - \gamma_{1,0,1} - \gamma_{1,\infty,1}}{1 - a_{2k-1,2}} + \gamma_{2k-2,0,1} + \gamma_{1,0,1} + \gamma_{1,\infty,1} - iT_1\right) + \right.$$
$$(\alpha_{2k-1,1,2} - i)\left(\frac{-\alpha_{2k-1,1,2} - \gamma_{1,0,2} - \gamma_{1,\infty,2} + i}{1 - a_{2k-1,2}} + \gamma_{2k-2,0,2} + \gamma_{1,0,2} + \gamma_{1,\infty,2} - iT_2\right) +$$
$$\left.\alpha_{2k-1,1,3}\left(\frac{-\alpha_{2k-1,1,3} - \gamma_{1,0,3} - \gamma_{1,\infty,3}}{1 - a_{2k-1,2}} + \gamma_{2k-2,0,3} + \gamma_{1,0,3} + \gamma_{1,\infty,3}\right)\right] + o(x).$$

When $x = 0$, the nonzero derivatives are

$$\frac{\partial}{\partial a_{2k-1,2}} \int_{C(a_{2k-1,1})} \frac{\mathcal{Q}}{dz} = 4\pi i \left[\frac{2}{(1 - a_{2k-1,2})^2}\right] = 2\pi i (T_2 + 1)^2.$$

The matrix of partial derivatives is invertible, and the statement follows from the implicit function theorem. □

Proposition 6.5. *For x close to 0, there exists unique $(\gamma_{2k-2,0,2}, \gamma_{2k-2,0,3})$ in a neighborhood of $(0,1)$, depending on the other parameters, such that*

$$\int_{C(a_{2k-1,2})} \frac{\mathcal{Q}}{dz} = 0$$

for $k = 2, 3, \dots, n$.

Proof. Using the expression for $\mathcal{Q}$ on $\overline{\mathbb{C}}_{2k-1}$ from the proof of Proposition 6.4,

$$\int_{C(a_{2k-1,2})} \frac{\mathcal{Q}}{dz}$$
$$= 4\pi i \left[(-\alpha_{2k-1,1,1} - \gamma_{1,0,1} - \gamma_{1,\infty,1})\left(\frac{\alpha_{2k-1,1,1}}{a_{2k-1,2} - 1} + \frac{\gamma_{2k-2,0,1} + \gamma_{1,0,1} + \gamma_{1,\infty,1} - iT_1}{a_{2k-1,2}}\right) + \right.$$
$$(-\alpha_{2k-1,1,2} - \gamma_{1,0,2} - \gamma_{1,\infty,2} + i)\left(\frac{\alpha_{2k-1,1,2} - i}{a_{2k-1,2} - 1} + \frac{\gamma_{2k-2,0,2} + \gamma_{1,0,2} + \gamma_{1,\infty_2} - iT_2}{a_{2k-1,2}}\right) +$$
$$\left.(-\alpha_{2k-1,1,3} - \gamma_{1,0,3} - \gamma_{1,\infty.3})\left(\frac{\alpha_{2k-1,1,3}}{a_{2k-1,2} - 1} + \frac{\gamma_{2k-2,0,3} + \gamma_{1,0,3} + \gamma_{1,\infty,3}}{a_{2k-1,2}}\right)\right] + o(x).$$

When $x=0$, the nonzero derivatives are

$$\frac{\partial}{\partial \gamma_{2k-2,0,2}} \int_{C(a_{2k-1,2})} \frac{\mathcal{Q}}{dz} = 4\pi i \left[\frac{i}{a_{2k-1,2}}\right] = -\frac{4\pi(T_2+1)}{T_2-1}$$
$$\frac{\partial}{\partial \gamma_{2k-2,0,3}} \int_{C(a_{2k-1,2})} \frac{\mathcal{Q}}{dz} = 4\pi i \left[\frac{1}{a_{2k-1,2}}\right] = \frac{4\pi i(T_2+1)}{T_2-1}.$$

The matrix of partial derivatives is invertible, and the statement follows from the implicit function theorem. ☐

Proposition 6.6. *For x close to 0, there exists unique $t_{2k-1,1}$ in a neighborhood of $t^0_{2k-1,1}$, depending on the other parameters, such that*

$$\int_{C(a_{2k-1,1})} (z-a_{2k-1,1})\frac{\mathcal{Q}}{dz} = 0, \quad k=1,\dots,n.$$

Proof. When $x=0$, by Lemma 3.4, the principal part of $\frac{\partial \phi_1}{\partial t_{2k-1,1}}$ at $a_{2k-1,1}$ is

$$\frac{-dz}{2\pi i(z-1)^2}\int_{C(b_{2k-1,1})} \frac{1}{z-1}\left(-\frac{iT_1}{z}\right) dz = \frac{iT_1 dz}{(z-1)^2},$$

the principal part of $\frac{\partial \phi_2}{\partial t_{2k-1,1}}$ at $a_{2k-1,1}$ is

$$\frac{-dz}{2\pi i(z-1)^2}\int_{C(b_{2k-1,1})} \frac{1}{z-1}\left(\frac{i}{z-1} - \frac{i}{z-b_{2k-1,2}} - \frac{iT_2}{z}\right) dz = \frac{i(T_2+1)dz}{2(z-1)^2},$$

and the principal part of $\frac{\partial \phi_3}{\partial t_{2k-1,1}}$ at $a_{2k-1,1}$ is

$$\frac{-dz}{2\pi i(z-1)^2}\int_{C(b_{2k-1,1})} \frac{1}{z-1}\left(-\frac{1}{z-1} - \frac{1}{z-b_{2k-1,2}} + \frac{1}{z}\right) dz = -\frac{(T_2+1)dz}{2(z-1)^2}.$$

Using the expression for $\mathcal{Q}$ on $\overline{\mathbb{C}}_{2k-1}$ from the proof of Proposition 6.4 when $x=0$,

$$
\begin{aligned}
&\frac{\partial}{\partial t_{2k-1,1}}\int_{C(a_{2k-1,1})}(z-1)\frac{\mathcal{Q}}{dz}\\
=&2\int_{C(a_{2k-1,1})}(z-1)\frac{\phi_1}{dz}\frac{\partial\phi_1}{\partial t_{2k-1,1}}+(z-1)\frac{\phi_2}{dz}\frac{\partial\phi_2}{\partial t_{2k-1,1}}+(z-1)\frac{\phi_3}{dz}\frac{\partial\phi_3}{\partial t_{2k-1,1}}\\
=&2\int_{C(a_{2k-1,1})}\left[\left(\frac{iT_1}{z-1}\right)\left(-\frac{iT_1}{z}\right)+\frac{i(T_2+1)}{2(z-1)}\left(\frac{-i}{z-1}+\frac{i}{z-a_{2k-1,2}}-\frac{iT_2}{z}\right)-\right.\\
&\qquad\left.\frac{T_2+1}{2(z-1)}\left(\frac{1}{z-1}+\frac{1}{z-a_{2k-1,2}}-\frac{1}{z}\right)\right]dz\\
&=4\pi i\left[T_1^2-\frac{T_2+1}{2(1-a_{2k-1,2})}+\frac{T_2(T_2+1)}{2}-\frac{T_2+1}{2(1-a_{2k-1,2})}+\frac{T_2+1}{2}\right]\\
&=4\pi i(1-T_2^2).
\end{aligned}
$$

The matrix of partial derivatives is invertible, and the statement follows from the implicit function theorem. □

Proposition 6.7. *For x close to 0, there exists unique $(\alpha_{2k-1,1,2},\alpha_{2k-1,1,3})$ in a neighborhood of $(0,1)$, depending on the other parameters, such that*

$$\int_{C(a_{2k-1,2})}(z-a_{2k-1,2})\frac{\mathcal{Q}}{dz}=0,\quad k=1,\ldots,n.$$

Proof. Using the expression for $\mathcal{Q}$ on $\overline{\mathbb{C}}_{2k-1}$ from the proof of Proposition 6.4,

$$
\begin{aligned}
\int_{C(a_{2k-1,2})}(z-a_{2k-1,2})\frac{\mathcal{Q}}{dz}=2\pi i\,\big[&(\gamma_{1,0,1}+\gamma_{1,\infty,1}+\alpha_{2k-1,1,1})^2+\\
&(\gamma_{1,0,2}+\gamma_{1,\infty,2}+\alpha_{2k-1,1,2}-i)^2+\\
&(\gamma_{1,0,3}+\gamma_{1,\infty,3}+\alpha_{2k-1,1,3})^2\big]+o(x).
\end{aligned}
$$

When $x=0$, the nonzero derivatives are

$$
\begin{aligned}
\frac{\partial}{\partial\alpha_{2k-1,1,2}}\int_{C(a_{2k-1,2})}(z-a_{2k-1,2})\frac{\mathcal{Q}}{dz}&=4\pi\\
\frac{\partial}{\partial\alpha_{2k-1,1,3}}\int_{C(a_{2k-1,2})}(z-a_{2k-1,2})\frac{\mathcal{Q}}{dz}&=-4\pi i.
\end{aligned}
$$

The matrix of partial derivatives is invertible, and the statement follows from the implicit function theorem. □

Proposition 6.8. *For x close to 0, there exists unique $(\gamma_{1,\infty,2}, T_2, \gamma_{2n,0,2}, \gamma_{2n,0,3})$ in a neighborhood of $(0, T_2^0, 0, 1)$, respectively, depending on the other parameters, such that*

$$\int_{C(0_{2n})} \frac{\mathcal{Q}}{dz} = \int_{C(0_{2n})} \frac{z\mathcal{Q}}{dz} = 0.$$

Proof. On $\overline{C}_{2n}$,

$$\mathcal{Q} = \Bigg[\left(-\frac{\alpha_{2n-1,1,1}}{z-1} + \frac{\alpha_{2n-1,1,1} + \gamma_{1,0,1} + \gamma_{1,\infty,1}}{z - b_{2n-1,2}} + \frac{\gamma_{2n,0,1} - iT_1}{z}\right)^2 + \left(\frac{-\alpha_{2n-1,1,2} + i}{z-1} + \frac{\alpha_{2n-1,1,2} + \gamma_{1,0,2} + \gamma_{1,\infty,2} - i}{z - b_{2n-1,2}} + \frac{\gamma_{2n,0,2} - iT_2}{z}\right)^2 + \left(-\frac{\alpha_{2n-1,1,3}}{z-1} + \frac{\alpha_{2n-1,1,3} + \gamma_{1,0,3} + \gamma_{1,\infty,3}}{z - b_{2n-1,2}} + \frac{\gamma_{2n,0,3}}{z}\right)^2\Bigg] dz^2 + o(x).$$

Then,

$$\int_{C(0_{2n})} \frac{\mathcal{Q}}{dz} = 4\pi i \Bigg[(\gamma_{2n,0,1} - iT_1)\left(\alpha_{2n-1,1,1} - \frac{\alpha_{2n-1,1,1} + \gamma_{1,0,1} + \gamma_{1,\infty,1}}{b_{2n-1,2}}\right) + (\gamma_{2n,0,2} - iT_2)\left(\alpha_{2n-1,1,2} - i - \frac{\alpha_{2n-1,1,2} - \gamma_{1,0,2} + \gamma_{1,\infty,2} - i}{b_{2n-1,2}}\right) + (\gamma_{2n,0,3})\left(\alpha_{2n-1,1,3} - \frac{\alpha_{2n-1,1,3} + \gamma_{1,0,3} + \gamma_{1,\infty,3}}{b_{2n-1,2}}\right)\Bigg] + o(x)$$

and

$$\int_{C(0_{2n})} \frac{z\mathcal{Q}}{dz} = 2\pi i \left[(\gamma_{2n,0,1} - iT_1)^2 + (\gamma_{2n,0,2} - iT_2)^2 + \gamma_{2n,0,3}^2\right] + o(x).$$

When $x = 0$,

$$\frac{\partial}{\partial \gamma_{1,\infty,2}} \int_{C(0_{2n})} \frac{\mathcal{Q}}{dz} = -\frac{4\pi T_2}{b_{2n-1,2}} = -\frac{4\pi T_2(T_2 - 1)}{T_2 + 1}$$

$$\frac{\partial}{\partial T_2} \int_{C(0_{2n})} \frac{\mathcal{Q}}{dz} = 4\pi i \left[-1 + \frac{1}{b_{2n-1,2}}\right] = -\frac{8\pi i}{T_2 + 1}$$

$$\frac{\partial}{\partial \gamma_{2n,0,2}} \int_{C(0_{2n})} \frac{\mathcal{Q}}{dz} = 4\pi \left[1 - \frac{1}{b_{2n-1,2}}\right] = \frac{8\pi}{T_2 + 1}$$

$$\frac{\partial}{\partial \gamma_{2n,0,3}} \int_{C(0_{2n})} \frac{\mathcal{Q}}{dz} = 4\pi i \left[1 + \frac{1}{b_{2n-1,2}}\right] = \frac{8\pi i T_2}{T_2 + 1}$$

and

$$\frac{\partial}{\partial \gamma_{1,\infty,2}} \int_{C(0_{2n})} \frac{zQ}{dz} = 0$$
$$\frac{\partial}{\partial T_2} \int_{C(0_{2n})} \frac{zQ}{dz} = -4\pi i T_2$$
$$\frac{\partial}{\partial \gamma_{2n,0,2}} \int_{C(0_{2n})} \frac{zQ}{dz} = 4\pi T_2$$
$$\frac{\partial}{\partial \gamma_{2n,0,3}} \int_{C(0_{2n})} \frac{zQ}{dz} = 4\pi i.$$

The matrix of partial derivatives is invertible, and the statement follows from the implicit function theorem. □

The remaining variables are $t_{1,2}$ and $\gamma_{2n,0,1}$. The parameter x means that $t_{k,i} \neq 0$ if $t_{1,2} \neq 0$. Hence, when $t_{1,2} \neq 0$, all of the nodes open as needed.

7. Embeddedness and Regularity

The immersion is regular if $|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 > 0$. Employing the technique used in [13], we split the surface into two pieces. Let $\Omega_{k,\delta}$ be the set of points in $\overline{\mathbb{C}}_k$ which are distance greater than delta away from the punctures $a_{k,i}, b_{k,+1,i}, 0_k, \infty_k$, and let

$$\Omega_\delta = \cup_{k=1}^m \Omega_{k,\delta}$$

When $x = 0$,

$$|\phi_3|^2 = \frac{1 - T_2^2}{|z|^2} > 0$$

on Ω_δ.

When $x = 0$, ϕ_1 has 2 zeros in $\Omega_{k,\delta}$, for small enough δ. By continuity, ϕ_1 has 2 zeros in $\Omega_{k,\delta}$, for small x. Thus, ϕ_1 has $2m$ zeros in Ω_δ. The genus of Σ_t is $g = m-1$, and ϕ_1 has four poles on Σ_t. Hence, ϕ_1 has $2m$ zeros on Σ_t, and all of the zeros of ϕ_1 are in Ω_δ. Thus, the surface is regular.

We can prove the surface is embedded by examining Ω_δ and its complement. When $x = 0$, the image of $\Omega_{k,\delta}$ is an embedded doubly periodic Scherk surface. Thus, for small x, the image of $\Omega_{k,\delta}$ is embedded.

If $z_k \in \overline{\mathbb{C}}_k$ and $z_{k+1} \in \overline{\mathbb{C}}_{k+1}$ then

$$\lim_{x \to 0} -x^2 \operatorname{Re} \int_{z_k}^{z_{k+1}} \phi_1 = u_k \neq 0$$

Hence, for small x, the images of $\Omega_{k,\delta}$ and $\Omega_{k+1,\delta}$ don't overlap. Therefore, the image of Ω_δ is embedded.

Near the punctures $a_{2k-1,i}$ and $b_{2k-1,i}$, for small x, the surface normal is approximately $(1,0,0)$ for $i=1$ and $(-1,0,0)$ for $i=2$. Near the punctures 0_k and ∞_k, for small x, the surface normal is approximately $(0,\sqrt{1-T_2^2},-T_2)$ and $(0,-\sqrt{1-T_2^2},T_2)$, respectively. Hence, the surface is embedded in a neighborhood of each puncture.

Now, we show that the ends do not intersect. Let $V_{1,0}$, $V_{1,\infty}$, $V_{2n,0}$, $V_{2n,\infty}$ represent the flux vectors at the respective ends 0_1, ∞_1, 0_{2n}, and ∞_{2n}. By conformality, the gauss map satisfies

$$g(z) = -\frac{\phi_1 + i\phi_2}{\phi_3} = \frac{\phi_3}{\phi_1 - i\phi_2}.$$

At the ends, this translates to

$$g(z) = -\frac{\operatorname{Res}(\phi_1,z) + i\operatorname{Res}(\phi_2,z)}{\operatorname{Res}(\phi_3,z)} = \frac{\operatorname{Res}(\phi_3,z)}{\operatorname{Res}(\phi_1,z) - i\operatorname{Res}(\phi_2,z)},$$

which implies that all of the flux vectors have length equal to $2\pi\sqrt{T_1^2+T_2^2}$ and are all orthogonal to the period vector $(2\pi T_1, 2\pi T_2, 0)$. Since the flux vectors must sum to 0 by the Residue theorem, the flux vectors form a 1-parameter family, where the parameter is the angle between any pair of the flux vectors. The signs of the third component of the flux at $x=0$ yields

$$(V_{1,0}, V_{1,\infty}) = (-V_{2n,0}, -V_{2n,\infty}) \quad \text{or} \quad (V_{1,0}, V_{1,\infty}) = (-V_{2n,\infty}, -V_{2n,0}).$$

For the the surface to be embedded, we require that the angle between $V_{1,0}$ and $V_{1,\infty}$ is 0. This introduces 1 more real equation, which can be written as

$$\gamma_{2n,0,1} = -\frac{1}{2}\left(\gamma_{1,0,1} + \gamma_{1,\infty,1}\right).$$

This, together with the propositions from sections 5 and 6, prove Theorem 1.1. That is, the embedded surfaces form a 2-parameter family.

8. Sketch of proof of Theorem 1.2

We use the exact same method used in the proof of Theorem 1.1, with the following differences.

(1) Instead of having ends at $0_1, \infty_1, 0_{2n}$, and ∞_{2n}, the pairs of points $(0_1, \infty_{2n})$ and $(\infty_1, 0_{2n})$ are identified, creating two extra nodes which are opened up in the usual manner. This creates two extra gluing parameters, $t_{2n,1}$ and $t_{2n,2}$. The constructed Riemann surface Σ_t has genus $2n+1$. See Figure 8.1

(2) Lemma 3.2 and Lemma 3.3 are different, taking into account the change in genus and lack of ends. Thus, there are slight changes in how the Weierstrass data is defined and the conformal equations that need to be solved.

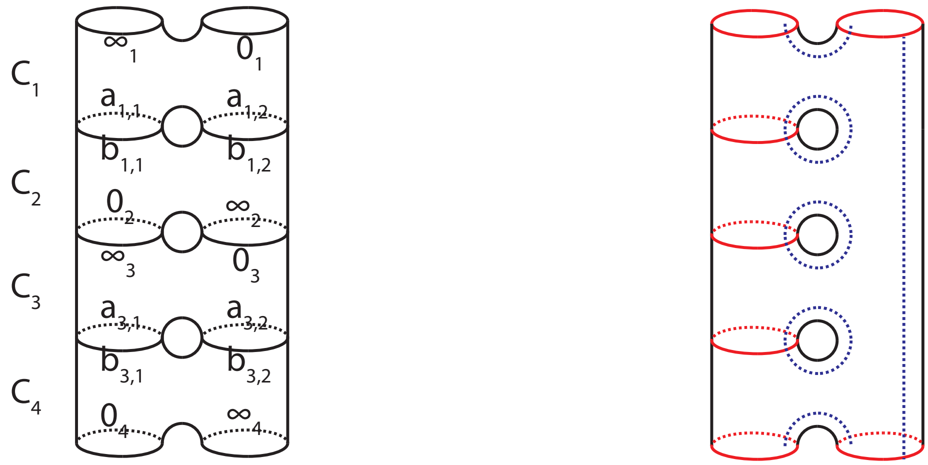

FIGURE 8.1. Models for Σ_t and its homology basis

Lemma 8.1. *For t close to 0, the map*

$$\omega \mapsto \left(\int_{C(0_1)} \omega, \int_{C(\infty_1)} \omega, \underbrace{\int_{C(0_{2k})} \omega}_{k \in \{1,\dots,n-1\}}, \underbrace{\int_{C(a_{2k-1,1})} \omega}_{k \in \{1,\dots,n\}} \right)$$

is an isomorphism from $\Omega^1(\Sigma_t)$ to $\mathbb{C}^g = \mathbb{C}^{2n+1}$.

Lemma 8.2. *For t close to 0, the map*

$$L(\psi) = \left(\underbrace{\int_{C(a_{2k-1,i})} \frac{(z - a_{2k-1,i})\psi}{dz}, \int_{C(a_{2k-1,i})} \frac{\psi}{dz}, \int_{C(0_{2k})} \frac{\psi}{dz}, \int_{C(b_{2k-1,2})} \frac{\psi}{dz}}_{k \in \{1,\dots,n\}, i \in \{1,2\}} \right)$$

is an isomorphism from $\Omega^2(\Sigma_t)$ to $\mathbb{C}^{3g-3} = \mathbb{C}^{6n}$.

(3) The only change in the definition of the Weierstrass data is that we don't define the residues at 0_{2n} Instead, the residues at 0_{2n} (∞_{2n} resp.) are the negative of the residues at ∞_1 (0_1 resp.).
(4) There are 2 extra periods, although only one needs to be solved. Thus, there are $6n$ period equations. The cycle $B_{2n,1}$ goes from $\overline{\mathbb{C}}_{2n}$ to $\overline{\mathbb{C}}_1$, passing through the neck corresponding to the node $(0_{2n}, \infty_1)$ and back to $\overline{\mathbb{C}}_{2n}$,

passing through the neck corresponding to the node $(\infty_{2n}, 0_1)$. We need

$$\mathrm{Re} \int_{B_{2n,1}} (\phi_1, \phi_2, \phi_3) = 0$$

The cycle B_1 runs through every copy of $\overline{\mathbb{C}}_k$, passing through the necks corresponding to the nodes $(a_{2k-1,2}, b_{2k-1,2})$, $(\infty_{2k}, 0_{2k+1})$, and $(\infty_{2n}, 0_1)$. This cycle is nonzero and corresponds to the third period of the surface.

(5) There are four extra real gluing parameters - we add $t_{2n,1}$, $t_{2n,2}$ - and three less period parameters - we lose $\gamma_{2n,0,1}, \gamma_{2n,0,2}, \gamma_{2n,0,3}$. Thus, there are $18n + 5$ real parameters and $18n$ real equations, leaving five free parameters. The free parameters are $T_1, T_2, t_{1,2}$, and $\gamma_{1,\infty,3}$.

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